

# A zero- $\frac{\sqrt{5}}{2}$ law for cosine families

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**Abstract :** Let  $a \in \mathbb{R}$ , and let  $k(a)$  be the largest constant such that  $\sup | \cos(na) - \cos(nb) | < k(a)$  for  $b \in \mathbb{R}$  implies that  $b \in \pm a + 2\pi\mathbb{Z}$ . We show that if a cosine sequence  $(C(n))_{n \in \mathbb{Z}}$  with values in a Banach algebra  $A$  satisfies  $\sup_{n \geq 1} \|C(n) - \cos(na).1_A\| < k(a)$ , then  $C(n) = \cos(na)$  for  $n \in \mathbb{Z}$ . Since  $\frac{\sqrt{5}}{2} \leq k(a) \leq \frac{8}{3\sqrt{3}}$  for every  $a \in \mathbb{R}$ , this shows that if some cosine family  $(C(g))_{g \in G}$  over an abelian group  $G$  in a Banach algebra satisfies  $\sup_{g \in G} \|C(g) - c(g)\| < \frac{\sqrt{5}}{2}$  for some scalar cosine family  $(c(g))_{g \in G}$ , then  $C(g) = c(g)$  for  $g \in G$ , and the constant  $\frac{\sqrt{5}}{2}$  is optimal. We also describe the set of all real numbers  $a \in [0, \pi]$  satisfying  $k(a) \leq \frac{3}{2}$ .

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**AMS classification :** *Primary 46J45, 47D09, Secondary 26A99*

## 1 Introduction

Let  $G$  be an abelian group. Recall that a  $G$ -cosine family of elements of a unital normed algebra  $A$  with unit element  $1_A$  is a family  $(C(g))_{g \in G}$  of elements of  $A$  satisfying the so-called d'Alembert equation

$$C_0 = 1_A, C(g+h) + C(g-h) = 2C(g)C(h) \quad (g \in G, h \in G). \quad (1)$$

A  $\mathbb{R}$ -cosine family is called a cosine function, and a  $\mathbb{Z}$ -cosine family is called a cosine sequence.

A cosine family  $C = (C(g))_{g \in G}$  is said to be bounded if there exists  $M > 0$  such that  $\|C(g)\| \leq M$  for every  $g \in G$ . In this case we set

$$\|C\|_\infty = \sup_{g \in G} \|C(g)\|, \text{dist}(C_1, C_2) = \|C_1 - C_2\|_\infty.$$

A cosine family is said to be scalar if  $C(g) \in \mathbb{C}.1_A$  for every  $g \in G$ . It is easy to see and well-known that a bounded scalar cosine sequence satisfies  $C(n) = \cos(an)$  for some  $a \in \mathbb{R}$ .

Strongly continuous operator valued cosine functions are a classical tool in the study of differential equations, see for example [2], [3], [15], [19], and a functional calculus approach to these objects was developped recently in [11].

Bobrowski and Chojnacki proved in [4] that if a strongly continuous operator valued cosine function on a Banach space  $(C(t))_{t \in \mathbb{R}}$  satisfies  $\sup_{t \geq 0} \|C(t) - c(t)\| < 1/2$  for some scalar bounded continuous cosine function  $c(t)$  then  $C(t) = c(t)$  pour  $t \in \mathbb{R}$ , and Zwart and F. Schwenninger showed in [18] that this result remains valid under the condition  $\sup_{t \geq 0} \|C(t) - c(t)\| < 1$ . The proofs were based on rather involved arguments from operator theory and semigroup theory. Very recently, Bobrowski, Chojnacki and Gregosiewicz [5] showed more precisely that if a cosine function  $C = C(t)$  satisfies  $\sup_{t \in \mathbb{R}} \|C(t) - c(t)\| < \frac{8}{3\sqrt{3}}$  for some scalar bounded continuous cosine function  $c(t)$ , then  $C(t) = c(t)$  for  $t \in \mathbb{R}$ , without any continuity assumption on  $C$ , and the same result was obtained independently by the author in [10]. The constant  $\frac{8}{3\sqrt{3}}$  is obviously optimal, since  $\sup_{t \in \mathbb{R}} |\cos(at) - \cos(3at)| = \frac{8}{3\sqrt{3}}$  for every  $a \in \mathbb{R} \setminus \{0\}$ .

The author also proved in [10] that if a cosine sequence  $(C(t))_{t \in \mathbb{R}}$  satisfies  $\sup_{t \in \mathbb{R}} \|C(t) - \cos(at)1_A\| = m < 2$  for some  $a \neq 0$ , then the closed algebra generated by  $(C(t))_{t \in \mathbb{R}}$  is isomorphic to  $\mathbb{C}^k$  for some  $k \geq 1$ , and that there exists a finite family  $p_1, \dots, p_k$  of pairwise orthogonal idempotents of  $A$  and a family  $(b_1, \dots, b_k)$  of distinct elements of the finite set  $\Delta(a, m) := \{b \geq 0 : \sup_{t \in \mathbb{R}} |\cos(bt) - \cos(at)| \leq m\}$  such that we have

$$C(t) = \sum_{j=1}^k \cos(b_j t) p_j \quad (j \in \mathbb{R}).$$

Also Chojnacki developped in [7] an elementary argument to show that if  $(C(n))_{n \in \mathbb{Z}}$  is a cosine sequence in a unital normed algebra  $A$  satisfying  $\sup_{n \geq 1} \|C(n) - c(n)\| < 1$  for some scalar cosine sequence  $(c(n))_{n \in \mathbb{Z}}$  then  $c(n) = C(n)$  for every  $n$ , which obviously implies the result of Zwart and F. Schwenninger. His approach is based on an elaborated adaptation of a very short elementary argument used by Wallen in [20] to prove an improvement of the classical Cox-Nakamura-Yoshida-Hirschfeld-Wallen theorem [8], [13], [16] which shows that if an element  $a$  of a unital normed algebra  $A$  satisfies  $\sup_{n \geq 1} \|a^n - 1\| < 1$ , then  $a = 1$ .

Applying this result to the cosine sequences  $C(ng)$  and  $c(ng)$  for  $g \in G$ , Chojnacki observed in [7] that if a cosine family  $C(g)$  satisfies  $\sup_{g \in G} \|C(g) - c(g)\| < 1$  for some scalar cosine family  $c(g)$  then  $C(g) = c(g)$  for every  $g \in G$ .

In the same direction Schwenninger and Zwart showed in [17] that if a cosine sequence  $(C(n))_{n \in \mathbb{Z}}$  in a Banach algebra  $A$  satisfies  $\sup_{n \geq 1} \|C(n) - 1_A\| < \frac{3}{2}$ , then  $C(n) = 1_A$  for every  $n$ .

The purpose of this paper is to obtain optimal results of this type. We prove

a "zero- $\frac{\sqrt{5}}{2}$ " law : if a cosine family  $(C(g))_{g \in G}$  satisfies  $\sup_{g \in G} \|C(g) - c(g)\| < \frac{\sqrt{5}}{2}$  for some scalar cosine family  $(c(g))_{g \in G}$  then  $C(g) = c(g)$  for every  $g \in G$ . Since  $\sup_{n \geq 1} |\cos(\frac{2n\pi}{5}) - \cos(\frac{4n\pi}{5})| = \cos(\frac{2\pi}{5}) - \cos(\frac{\pi}{5}) = \frac{\sqrt{5}}{2}$ , the constant  $\frac{\sqrt{5}}{2}$  is optimal.

In fact for every  $a \in \mathbb{R}$  there exists a largest constant  $k(a)$  such that  $\sup_{n \geq 1} |\cos(nb) - \cos(na)| < k(a)$  implies that  $\cos(nb) = \cos(na)$  for  $n \geq 1$ , and we prove that if a cosine sequence  $(C(n))_{n \in \mathbb{Z}}$  in a Banach algebra  $A$  satisfies  $\sup_{n \geq 1} |C(n) - \cos(na)1_A| < k(a)$  then  $C(n) = \cos(na)$  for  $n \geq 1$ . This follows from the following result, proved by the author in [10].

**Theorem 1.1.** *Let  $(C(n))_{n \in \mathbb{Z}}$  be a bounded cosine sequence in a Banach algebra. If  $\text{spec}(C(1))$  is a singleton, then the sequence  $(C(n))_{n \in \mathbb{Z}}$  is scalar, and so there exists  $a \in \mathbb{R}$  such that  $C(n) = \cos(na)$  for  $n \geq 1$ .*

The second part of the paper is devoted to a discussion of the values of the constant  $k(a)$ . As mentioned above, it follows from [17] that  $k(0) = \frac{3}{2}$ , and it is obvious that  $k(a) \leq \sup_{n \geq 1} |\cos(na) - \cos(3na)| \leq \frac{8}{3\sqrt{3}}$  if  $a \notin \frac{\pi}{2}\mathbb{Z}$ . We observe that  $k(a) = \frac{8}{3\sqrt{3}}$  if  $\frac{a}{\pi}$  is irrational, and we prove, using basic results about cyclotomic fields, that  $k(a) < \frac{8}{3\sqrt{3}}$  if  $\frac{a}{\pi}$  is rational.

We also show that the set  $\Omega(m) := \{a \in [0, \pi] \mid k(a) \leq m\}$  is finite for every  $m < \frac{8}{3\sqrt{3}}$ . We describe in detail the set  $\Omega(\frac{3}{2})$  : it contains 43 elements, and the only values for  $k(a)$  for which  $k(a) < \frac{3}{2}$  are  $\frac{\sqrt{2}}{5} = \cos(\frac{\pi}{5}) + \cos(\frac{2\pi}{5}) \approx 1.1180$ ,  $\sqrt{2} = \cos(\frac{\pi}{4}) + \cos(\frac{3\pi}{4}) \approx 1.4142$ , and  $\cos(\frac{2\pi}{11}) + \cos(\frac{3\pi}{11}) \approx 1.4961$ .

The zero- $\frac{\sqrt{5}}{2}$  law follows then from the fact that  $k(a) \geq \cos(\frac{\pi}{5}) + \cos(\frac{\pi}{5}) = \frac{\sqrt{5}}{2}$  for every  $a \in \mathbb{R}$ .

We also show that given  $a \in \mathbb{R}$  and  $m < 2$  the set  $\Gamma(a, m)$  of scalar cosine sequences  $(c(n))_{n \in \mathbb{Z}}$  satisfying  $\sup_{n \in \mathbb{Z}} |c(n) - \cos(na)| \leq m$  is finite. This implies that if a cosine sequence  $(C(n))_{n \in \mathbb{Z}}$  satisfies  $\sup_{n \in \mathbb{Z}} \|C(n) - \cos(na)1_A\| \leq m$ , then there exists  $k \leq \text{card}(\Gamma(a, m))$  such that the closed algebra generated by  $(C(n))_{n \in \mathbb{Z}}$  is isomorphic to  $\mathbb{C}^k$ , and there exists a finite family  $p_1, \dots, p_k$  of pairwise orthogonal idempotents of  $A$  and a finite family  $c_1, \dots, c_k$  of distinct elements of  $\Gamma(a, m)$  such that we have

$$C(n) = \sum_{j=1}^k c_j(n) p_j \quad (n \in \mathbb{Z}).$$

This last result does not extend to cosine families over general abelian group. Let  $G = (\mathbb{Z}/3\mathbb{Z})^{\mathbb{N}}$  : we give an easy example of a  $G$ -cosine family  $(C(g))_{g \in G}$  with values in  $l^\infty$  such that the closed subalgebra generated by  $(C(g))_{g \in G}$  equals  $l^\infty$ , while  $\sup_{g \in G} \|1_{l^\infty} - C(g)\| = \frac{3}{2}$ .

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## 2 Distance between bounded scalar cosine sequences

We introduce the following notation, to be used throughout the paper.

**Definition 2.1.** Let  $a \in \pi\mathbb{Q}$ . The order of  $a$ , denoted by  $\text{ord}(a)$ , is the smallest integer  $u \geq 1$  such that  $e^{iua} = 1$ .

Recall that a subset  $S$  of the unit circle  $\mathbb{T}$  is said to be independent if  $z_1^{n_1} \dots z_k^{n_k} \neq 1$  for every finite family  $(z_1, \dots, z_k)$  of distincts elements of  $S$  and every family  $(n_1, \dots, n_k) \in \mathbb{Z}^k$  of such that  $z_j \neq 0$  for  $j \leq k$ . It follows from a classical theorem of Kronecker, see for example [14], page 21 that if  $S = \{z_1, \dots, z_k\}$  is a finite independent set then the sequence  $(z_1^n, \dots, z_k^n)_{n \geq 1}$  is dense in  $\mathbb{T}^k$ . We deduce from Kronecker's theorem the following observation.

**Proposition 2.2.** Let  $a \in [0, \pi]$ . For  $m \geq 0$ , set

$$\Gamma(a, m) = \{b \in [0, \pi] : \sup_{n \geq 1} |\cos(na) - \cos(nb)| \leq m\}.$$

Then  $\Gamma(a, m)$  is finite for every  $m < 2$ .

Proof : Fix  $m \in [1, 2)$ . Notice that if  $b \in \mathbb{R}$ , and if the set  $\{e^{ia}, e^{ib}\}$  is independent, then it follows from Kronecker's theorem that the sequence  $((e^{ina}, e^{inb}))_{n \geq 1}$  is dense in  $\mathbb{T}^2$ , and so  $\sup_{n \geq 1} |\cos(na) - \cos(nb)| = 2$ , and  $b \notin \Gamma(a, m)$ .

Suppose that  $\frac{a}{\pi} \in \mathbb{Q}$ , and denote by  $u$  the order of  $a$ , so that  $e^{iua} = 1$ . If  $\frac{b}{\pi} \notin \mathbb{Q}$ , then the sequence  $(e^{iunb})_{n \geq 1}$  is dense in  $\mathbb{T}$ , and so

$$2 \geq \sup_{n \geq 1} |\cos(na) - \cos(nb)| \geq \sup_{n \geq 1} |1 - \cos(nub)| = 2,$$

which shows that  $b \notin \Gamma(a, m)$ .

The same argument shows that if  $\frac{a}{\pi} \notin \mathbb{Q}$ , and if  $\frac{b}{\pi} \in \mathbb{Q}$ , then  $b \notin \Gamma(a, m)$ . So we are left with two situations

- 1)  $\frac{a}{\pi} \notin \mathbb{Q}$ , and there exists  $p \neq 0$ ,  $q \neq 0$  and  $k \in \mathbb{Z}$  such that  $bq = ap + 2k\pi$ .
- 2)  $\frac{a}{\pi} \in \mathbb{Q}$  and  $\frac{b}{\pi} \in \mathbb{Q}$ .

We consider the first case. Replacing  $b \in [0, \pi]$  by  $-b \in [-\pi, 0]$  if necessary we can assume that  $p \geq 1$  and  $q \geq 1$ , and we can assume that we have

$$qb = pa + \frac{2k\pi}{r},$$

with  $\gcd(p, q) = 1, r \geq 1, \gcd(r, k) = 1$  if  $k \neq 0$ .

We have, since  $\frac{ra}{\pi} \notin \mathbb{Q}$ ,

$$\begin{aligned} \sup_{n \geq 1} |\cos(na) - \cos(nb)| &\geq \sup_{n \geq 1} |\cos(nrqa) - \cos(nrqb)| \\ &= \sup_{n \geq 1} |\cos(nrqa) - \cos(nrpa)| = \sup_{t \in \mathbb{R}} |\cos(qt) - \cos(pt)|, \end{aligned}$$

Since  $\gcd(p, q) = 1$ , we have  $\sup_{t \in \mathbb{R}} |\cos(qt) - \cos(pt)| = 2$  if  $p$  or  $q$  is even, so we can assume that  $p$  and  $q$  are odd. Set  $s = \frac{q-1}{2}$ .

It follows from Bezout's identity that there exist  $n \geq 1$  such that  $e^{\frac{2in\pi}{q}} = e^{\frac{2is\pi}{q}}$  and setting  $t = \frac{2n\pi}{q}$ , we obtain

$$\sup_{t \in \mathbb{R}} |\cos(qt) - \cos(pt)| \geq 1 - \cos\left(\frac{2s\pi}{2s+1}\right) = 1 + \cos\left(\frac{\pi}{q}\right).$$

The same argument shows that we have

$$\sup_{t \in \mathbb{R}} |\cos(qt) - \cos(pt)| \geq 1 + \cos\left(\frac{\pi}{p}\right).$$

We obtain

$$p \leq \frac{\pi}{\arccos(m-1)}, q \leq \frac{\pi}{\arccos(m-1)}.$$

We also have

$$\begin{aligned} \sup_{n \geq 1} |\cos(na) - \cos(nb)| &\geq \sup_{n \geq 1} |\cos(nqa) - \cos(nqb)| \\ &= \sup_{n \geq 1} \left| \cos(nqa) - \cos\left(npa + \frac{2nkq\pi}{r}\right) \right|. \end{aligned}$$

Assume that  $k \neq 0$ , set  $d = \gcd(r, q), r_1 = \frac{r}{d}, q_1 = \frac{q}{d}$ . Then  $\gcd(kq_1, r_1) = 1$ , and so there exists  $u \geq 1$  such that  $\frac{2ukq_1\pi}{r_1} \in \frac{2\pi}{r_1} + 2\pi\mathbb{Z}$ . This gives

$$\sup_{n \geq 1} |\cos(na) - \cos(nb)| \geq \sup_{n \geq 1} \left| \cos(nuqa) - \cos\left(npua + \frac{2n\pi}{r_1}\right) \right|.$$

If  $r_1$  is even, set  $r_2 = \frac{r_1}{2}$ . We obtain

$$\begin{aligned} &\sup_{n \geq 1} \left| \cos(nuqa) - \cos\left(npua + \frac{2n\pi}{r_1}\right) \right| \\ &\geq \sup_{n \geq 0} |\cos((2n+1)r_2uqa) - \cos((2n+1)r_2upa + \pi)|. \end{aligned}$$

Since  $2r_2ua \notin \pi\mathbb{Q}$ , there exists a sequence  $(n_j)_{j \geq 1}$  of integers such that

$$\lim_{j \rightarrow +\infty} \left| e^{i2n_j r_2 ua + i r_2 ua} \right| = 1,$$

so that

$$\lim_{j \rightarrow +\infty} \left| \cos((2n_j + 1)r_2 uqa) - \cos((2n_j + 1)r_2 upa) + \pi \right| = 2,$$

and in this situation  $\sup_{n \geq 1} |\cos(na) - \cos(nb)| = 2$ .

So we can assume that  $r_1$  is odd. Set  $r_2 = \frac{r_1 - 1}{2}$ . The same calculation as above gives

$$\begin{aligned} & \sup_{n \geq 1} \left| \cos(nuqa) - \cos\left(npua + \frac{2n\pi}{r_1}\right) \right| \\ & \geq \sup_{n \geq 1} \left| \cos((n(2r_2 + 1) + r_2)uqa) - \cos\left((n(2r_2 + 1) + r_2)upa + \frac{2(n(2r_2 + 1) + r_2)}{2r_2 + 1}\pi\right) \right| \\ & \geq 1 + \cos\left(\frac{\pi}{2r_2 + 1}\right). \end{aligned}$$

$$\text{Hence } r_1 = 2r_2 + 1 \leq \frac{\pi}{\arccos(m-1)}, r = r_1 d \leq r_1 q \leq \left(\frac{\pi}{\arccos(m-1)}\right)^2.$$

This gives

$$2|k|\pi \leq r|qb - pa| \leq 2\pi \left(\frac{\pi}{\arccos(m-1)}\right)^3, |k| \leq \left(\frac{\pi}{\arccos(m-1)}\right)^3.$$

We see that  $\Gamma(a, m)$  is finite if  $\frac{a}{\pi} \notin \mathbb{Q}$ , and that we have

$$\text{card}(\Gamma(a, m)) \leq 2 \left(\frac{\pi}{\arccos(m-1)}\right)^7.$$

Now consider the case where  $\frac{a}{\pi} \in \mathbb{Q}$ ,  $\frac{b}{\pi} \in \mathbb{Q}$ . We first discuss the case where  $a = 0, b \neq 0$ . We have  $b = \frac{p\pi}{q}$ , where  $1 \leq p \leq q, \gcd(p, q) = 1$

If  $p = q = 1$ , then  $b = \pi$ , and  $\sup_{n \geq 1} |1 - \cos(n\pi)| = 2$ . So we may assume that  $p \leq q - 1$ . If  $p$  is odd, then we have

$$\sup_{n \geq 1} |1 - \cos(nb)| \geq |1 - \cos(qb)| = 1 - \cos(p\pi) = 2.$$

So we can assume that  $p$  is even, so that  $q$  is odd. Set  $r = \frac{q-1}{2}$ . There exists  $n_0 \geq 1$  and  $r \in \mathbb{Z}$  such that  $n_0 p - r \in q\mathbb{Z}$ , and we have

$$\sup_{n \in \mathbb{Z}} |1 - \cos(nb)| \geq |1 - \cos(2n_0 b)| = \left| 1 - \cos\left(\frac{2r\pi}{2r+1}\right) \right| = 1 + \cos\left(\frac{\pi}{q}\right).$$

$$\text{We obtain again } q \leq \frac{\pi}{\arccos(m-1)}, \text{ and } \text{card}(\Gamma(0, m)) \leq \left(\frac{\pi}{\arccos(m-1)}\right)^2.$$

Now assume that  $a \neq 0$ , and let  $u \geq 2$  be the order of  $a$ . We have

$$\sup_{n \geq 1} |1 - \cos(nub)| = \sup_{n \geq 1} |\cos(nua) - \cos(nub)| \leq m,$$

and so there exists  $c \in \Gamma(0, m)$  such that  $\cos(nc) = \cos(nub)$  for  $n \geq 1$ . In particular  $\cos(c) = \cos(ub)$ , and  $b = \pm \frac{c}{u} + \frac{2k\pi}{u}$ , where  $k \in \mathbb{Z}$ . We obtain

$$\text{card}(\Gamma(a, m)) \leq 2u \text{card}(\Gamma(0, m)) \leq 2u \text{card}\left(\frac{\pi}{\arccos(m-1)}\right)^2.$$

□

We do not know whether it is possible to obtain a majorant for  $\text{card}(\Gamma(a, m))$  which depends only on  $m$  when  $a \in \pi\mathbb{Q}$ .

**Theorem 2.3.** *Let  $a \in \mathbb{R}$ , let  $m < 2$ , and let  $(C(n))_{n \in \mathbb{Z}}$  be a cosine sequence in a Banach algebra  $A$  such that  $\sup_{n \geq 1} \|C(n) - \cos(na)\| \leq m$ . Then there exists  $k \leq \text{card}(\Gamma(a, m))$  such that the closed algebra generated by  $(C(n))_{n \in \mathbb{Z}}$  is isomorphic to  $\mathbb{C}^k$ , and there exists a finite family  $p_1, \dots, p_k$  of pairwise orthogonal idempotents of  $A$  and a finite family  $b_1, \dots, b_k$  of distinct elements of  $\Gamma(a, m)$  such that we have*

$$C(n) = \sum_{j=1}^k \cos(nb_j) p_j \quad (n \in \mathbb{Z}).$$

Proof: Since  $c_n = P_n(c_1)$ , where  $P_n$  denotes the  $n$ -th Tchebishev polynomial,  $A_1$  is the closed unital subalgebra generated by  $c_1$  and the map  $\chi \rightarrow \chi(c_1)$  is a bijection from  $\widehat{A_1}$  onto  $\text{spec}_{A_1}(c_1)$ . Now let  $\chi \in \widehat{A_1}$ . The sequence  $(\chi(c_n))_{n \geq 1}$  is a scalar cosine sequence, and we have

$$\sup_{n \geq 1} |\cos(na) - \chi(c_n)| < 2.$$

It follows then from proposition 2.2 that  $\text{spec}_{A_1}(c_1) := \{\lambda = \chi(c_1) : \chi \in \widehat{A_1}\}$  is finite. Hence  $\widehat{A_1}$  is finite. Let  $\chi_1, \dots, \chi_m$  be the elements of  $\widehat{A_1}$ . It follows from the standard one-variable holomorphic functional calculus, see for example [9], that there exists for every  $j \leq m$  an idempotent  $p_j$  of  $A_1$  such that  $\chi_j(p_j) = 1$  and  $\chi_k(p_j) = 0$  for  $k \neq j$ . Hence  $p_j p_k = 0$  for  $j \neq k$ , and  $\sum_{j=1}^m p_j$  is the unit element of  $A_1$ .

Let  $x \in A_1$ . Then  $(p_j c_n)_{n \in \mathbb{Z}}$  is a cosine sequence in the commutative unital Banach algebra  $p_j A_1$ , and  $\text{spec}_{p_j A_1}(p_j c_1) = \{\chi_j(c_1)\}$ .

Since  $\sup_{n \geq 1} \|p_j \cos(na) - p_j c_n\| \leq 2\|p_j\|$ , the sequence  $(p_j c_n)_{n \geq 1}$  is bounded, and it follows from theorem 2.3 that  $(p_j c_n)_{n \geq 1}$  is a scalar sequence, and there exists  $\beta_j \in [0, \pi]$  such that  $p_j c_n = \chi_j(c_n) p_j = \cos(n\beta_j) p_j$  for  $n \in \mathbb{Z}$ .

Hence  $c_n = \sum_{j=1}^m \chi_j(c_n) p_j = \sum_{j=1}^m \cos(n\beta_j) p_j$  for  $n \geq 1$ . Since  $A_1$  is the closed subalgebra of  $A$  generated by  $c_1$ , we have  $x = \sum_{j=1}^m \chi_j(x) p_j$  for every  $x \in A_1$ , which shows that  $A_1$  is isomorphic to  $\mathbb{C}^m$ .  $\square$

**Corollary 2.4.** *Let  $a \geq 0 \in \mathbb{R}$ , and let  $k(a)$  be the largest positive real number  $m$  such that  $\Gamma(a, m) = \{a\}$  for every  $m < k(a)$ . If  $(C(n))_{n \in \mathbb{Z}}$  is a cosine sequence in a Banach algebra  $A$  such that  $\sup_{n \geq 1} \|C(n) - \cos(na)1_A\| < k(a)$ , then  $C(n) = \cos(na)1_A$  for  $n \in \mathbb{Z}$ .*

Theorem 2.3 does not extend to cosine families over general abelian groups, as shown by the following easy result.

**Proposition 2.5.** *Let  $G := (\mathbb{Z}/3\mathbb{Z})^{\mathbb{N}}$ . Then there exists a  $G$ -cosine family  $(C(g))_{g \in G}$  with values in  $l^\infty$  which satisfies the two following conditions*

- (i)  $\sup_{g \in G} \|1_{l^\infty} - C(g)\| = \frac{3}{2}$ ,
- (ii) *The algebra generated by the family  $(C(g))_{g \in G}$  is dense in  $l^\infty$ .*

Proof : Elements  $g$  of  $G$  can be written under the form  $g = (\overline{g}_m)_{m \geq 1}$ , where  $g_m \in \{0, 1, 2\}$ . Set

$$C(g) := \left( \cos\left(\frac{2g_m\pi}{3}\right) \right)_{m \geq 1}.$$

Then  $(C(g))_{g \in G}$  is a  $G$ -cosine family with values in  $l^\infty$  which obviously satisfies (i) since  $\cos\left(\frac{2\pi}{3}\right) = \cos\left(\frac{4\pi}{3}\right) = -\frac{1}{2}$ .

Now let  $\phi = (\phi_m)_{m \in \mathbb{Z}}$  be an idempotent of  $l^\infty$ , and let  $S := \{m \geq 1 \mid \phi_m = 1\}$ . Set  $g_m = 1$  if  $m \in S$ ,  $g_m = 0$  if  $m \geq 1, m \notin S$ , and set  $g = (\overline{g}_m)_{m \geq 1}$ . We have

$$C(0_G) - C(g) = 1_{l^\infty} - C(g) = \frac{3}{2}\phi,$$

and so  $\phi \in A$ . We can identify  $l^\infty$  to  $\mathcal{C}(\beta\mathbb{N})$ , the algebra of continuous functions on the Stone-Ćech compactification of  $\mathbb{N}$ , and  $\beta\mathbb{N}$  is an extremely disconnected compact set, which means that the closure of every open set is open, see for example [1], chap. 6, sec. 6. Since the characteristic function of every open and closed subset of  $\beta\mathbb{N}$  is an idempotent of  $l^\infty$ , the idempotents of  $l^\infty$  separate points of  $\beta\mathbb{N}$ , and it follows from the Stone-Weierstrass theorem that  $A$  is dense in  $l^\infty$ , which proves (ii).  $\square$



### 3 The values of the constant $k(a)$

It was shown in [17] that  $k(0) = \frac{3}{2}$ . We also have the following result.

**Proposition 3.1.** *We have  $k(a) = \frac{8}{3\sqrt{3}}$  if  $\frac{a}{\pi}$  is irrational, and  $k(a) < \frac{8}{3\sqrt{3}}$  if  $\frac{a}{\pi}$  is rational.*

Proof : Assume that  $\frac{a}{\pi} \notin \mathbb{Q}$ . Then  $3a \notin \pm a + 2\pi\mathbb{Z}$ , and we have

$$k(a) \leq \sup_{n \geq 1} |\cos(na) - \cos(3na)| = \sup_{x \in \mathbb{R}} |\cos(x) - \cos(3x)| = \frac{8}{3\sqrt{3}}.$$

We saw above that If  $\frac{b}{\pi}$  in  $\mathbb{Q}$ , then  $\sup_{n \geq 1} |\cos(na) - \cos(nb)| = 2$ , and we also have  $\sup_{n \geq 1} |\cos(na) - \cos(nb)| = 2$  if  $pa - qb \notin 2\pi\mathbb{Z}$  for  $(p, q) \neq (0, 0)$ . So if  $\sup_{n \geq 1} |\cos(na) - \cos(nb)| < 2$ , there exists  $p \in \mathbb{Z} \setminus \{0\}$ ,  $q \in \mathbb{Z} \setminus \{0\}$  and  $r \in \mathbb{Z}$  such that  $pa - qb = 2r\pi$ .

If  $p \neq \pm q$  then it follows from lemma 3.5 of [10] that we have

$$\begin{aligned} \sup_{n \geq 1} |\cos(na) - \cos(nb)| &\geq \sup_{n \geq 1} |\cos(nqa) - \cos(nqb)| = \sup_{n \geq 1} |\cos(qna) - \cos(pna)| \\ &= \sup_{x \in \mathbb{R}} |\cos(qx) - \cos(px)| = \sup_{x \in \mathbb{R}} \left| \cos\left(\frac{p}{q}x\right) - \cos(x) \right| \geq \frac{8}{3\sqrt{3}}. \end{aligned}$$

We are left with the case where  $b = \pm a + \frac{2s\pi}{r}$ , where  $r \in \mathbb{Z} \setminus \{-1, 0, 1\}$ , and we can restrict attention to the case where  $b = a + \frac{2s\pi}{r}$  where  $r \geq 2$ ,  $1 \leq s \leq r-1$ ,  $\gcd(r, s) = 1$ . It follows from Bezout's identity that there exists for every  $p \geq 1$  some  $u \in \mathbb{Z}$  such that  $ub - ua - \frac{2p\pi}{r} \in 2\pi\mathbb{Z}$ . If  $r$  is even, set  $p = \frac{r}{2}$ . We have, since the set  $\{e^{i(2n+1)a}\}_{n \geq 1}$  is dense in the unit circle,

$$\begin{aligned} \sup_{n \geq 1} |\cos(nb) - \cos(na)| &= \sup_{n \in \mathbb{Z}} |\cos(nb) - \cos(na)| \\ &\geq \sup_{n \geq 1} |\cos((2n+1)ub) - \cos((2n+1)ua)| \\ &= 2 \sup_{n \geq 1} |\cos((2n+1)ua)| = 2. \end{aligned}$$

Now assume that  $r$  is odd, and set  $p = \frac{r-1}{2}$ . We have

$$\begin{aligned} \sup_{n \geq 1} |\cos(nb) - \cos(na)| &\geq \sup_{n \geq 1} |\cos((2n+1)ub) - \cos((2n+1)ua)| \\ &\geq \sup_{n \geq 1} \left| \cos((2nr+1)ua) - \cos\left((2nr+1)ua + (2nr+1)\left(\pi - \frac{\pi}{r}\right)\right) \right| \end{aligned}$$

$$\geq \sup_{x \in \mathbb{R}} \left| \cos(x) + \cos\left(x - \frac{\pi}{r}\right) \right| \geq 2 \cos\left(\frac{\pi}{2r}\right) \geq \sqrt{3} > \frac{8}{3\sqrt{3}}.$$

Now assume that  $\frac{a}{\pi}$  is rational. If the order of  $a$  is equal to 1, then  $k(a) = 1.5$ , and we will see later that this is also true if the order of  $a$  equals 2 or 4.

Otherwise we have

$$k(a) \leq \sup_{n \geq 1} |\cos(na) - \cos(3na)| = \max_{1 \leq n \leq u} |\cos(na) - \cos(3na)|.$$

We have  $|\cos(nx) - \cos(3nx)| < \frac{8}{\pi\sqrt{3}}$  if  $x \notin \pm \arccos\left(\frac{1}{\sqrt{3}}\right) + \pi\mathbb{Z}$ . If  $na \in \pm \arccos\left(\frac{1}{\sqrt{3}}\right) + \pi\mathbb{Z}$  for some  $n \geq 1$ , then  $\frac{\arccos\left(\frac{1}{\sqrt{3}}\right)}{\pi}$  would be rational, and  $\alpha := \frac{1}{\sqrt{3}} + \frac{\sqrt{2}i}{\sqrt{3}}$  would be a root of unity. So  $\beta = \alpha^2 = -\frac{1}{3} + \frac{2\sqrt{2}i}{3}$  would have the form  $\beta = e^{\frac{2ik\pi}{n}}$  for some  $n \leq 1$  and some positive integer  $k \geq n$  such that  $\gcd(k, n) = 1$ .

Let  $\mathbb{Q}(\beta)$  be the smallest subfield of  $\mathbb{C}$  containing  $\mathbb{Q} \cup \beta$ . Since  $3\beta^2 + 2\beta + 3 = 0$ , the degree of  $\mathbb{Q}(\beta)$  over  $\mathbb{Q}$  is equal to 2. On the other hand the Galois group  $\text{Gal}(\mathbb{Q}(\beta)/\mathbb{Q})$  is isomorphic to  $(\mathbb{Z}/n\mathbb{Z})^\times$ , the group of invertible elements of  $\mathbb{Z}/n\mathbb{Z}$ , and we have, see [21], theorem 2.5

$$H(n) = \deg(\mathbb{Q}(\beta)/\mathbb{Q}) = 2,$$

where  $H(n) = \text{card}((\mathbb{Z}/n\mathbb{Z})^\times)$  denotes the number of integers  $p \in \{1, \dots, n\}$  such that  $\gcd(p, n) = 1$ .

Let  $P(n)$  be the set of prime divisors of  $n$ . It is well-known that we have, writing  $n = \prod_{p \in P(n)} p^{\alpha_p}$ , see for example [21], exercise 1.1,

$$H(n) = \prod_{p \in P(n)} p^{\alpha_p - 1} (p - 1).$$

It follows immediately from this identity that the only possibilities to get  $H(n) = 2$  are  $n = 3$ ,  $n = 4$ , and  $n = 6$ . Since  $\beta^3 \neq 1$ ,  $\beta^4 \neq 1$ , and  $\beta^6 \neq 1$ , we see that  $\frac{\beta}{\pi}$  is irrational, and so  $k(a) < \frac{8}{3\sqrt{3}}$  if  $\frac{a}{\pi}$  is rational.  $\square$

We know that if  $\frac{a}{\pi}$  is rational, and if  $\frac{b}{\pi}$  is irrational, then  $\sup_{n \geq 1} |\cos(na) - \cos(nb)| = 2$ . We discuss now the case where  $\frac{a}{\pi}$  and  $\frac{b}{\pi}$  are both rational, with  $b \notin \pm a + 2\pi\mathbb{Z}$ .

**Lemma 3.2.** *Let  $a, b \in (0, \pi]$ .*

(i) *If  $7a \leq b \leq \frac{\pi}{2}$ , or if  $\frac{\pi}{2} \leq b \leq \frac{5\pi}{6}$ , with  $|b - \frac{2\pi}{3}| \geq 7a$ , then*

$$\sup_{n \geq 1} |\cos(na) - \cos(nb)| > 1.55.$$

(ii) *If  $\frac{5\pi}{6} \leq b \leq \pi$ , and if  $b \geq 4a$ , then*

$$\cos(a) - \cos(b) > 1.57.$$

Proof : (i) Assume that  $7a \leq b \leq \frac{\pi}{2}$ , let  $p$  be the largest integer such that  $pb < \frac{3\pi}{4}$ , and set  $q = p + 1$ . We have  $\frac{3\pi}{4} \leq qb \leq \frac{5\pi}{4}$ ,  $0 \leq qa \leq \frac{5\pi}{28}$ , and we obtain

$$\sup_{n \geq 1} |\cos(na) - \cos(nb)| \geq \cos(qa) - \cos(qb) \geq \cos\left(\frac{5\pi}{28}\right) + \cos\left(\frac{\pi}{4}\right) > 1.55.$$

Now assume that  $\frac{\pi}{2} \leq b \leq \frac{5\pi}{6}$ , with  $|b - \frac{2\pi}{3}| \geq 7a$ , and set  $c = |3b - 2\pi|$ . Since  $|b - \frac{2\pi}{3}| \leq \frac{\pi}{6}$ , we have  $21a \leq c \leq \frac{\pi}{2}$ , and we obtain

$$\begin{aligned} \sup_{n \geq 1} |\cos(na) - \cos(nb)| &\geq \sup_{n \geq 1} |\cos(3na) - \cos(3nb)| \\ &= \sup_{n \geq 1} |\cos(3na) - \cos(nc)| > 1.55. \end{aligned}$$

(ii) If  $\frac{5\pi}{6} \leq b \leq \pi$ , and if  $b \geq 4a$ , then  $0 < a \leq \frac{\pi}{4}$ , and we have

$$\cos(a) - \cos(b) \geq \cos\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{6}\right) > 1.57.$$

**Lemma 3.3.** *Let  $p, q$  be two positive integers such that  $p < q$ .*

*(i) If  $q \neq 3p$ , then there exists  $u_{p,q} \geq 1$  such that, if  $\text{ord}(a) \geq u_{p,q}$  we have*

$$\sup_{n \geq 1} |\cos(npa) - \cos(nqa)| > \frac{8}{\sqrt{3}}.$$

*(ii) If  $q = 3p$ , then for every  $m < \frac{8}{3\sqrt{3}}$  there exists  $u_p(m) \geq 1$  such that if  $\text{ord}(a) \geq u_p(m)$  we have*

$$\sup_{n \geq 1} |\cos(npa) - \cos(3npa)| > m.$$

Proof : Set  $\lambda = \sup_{x \in \mathbb{R}} |\cos(px) - \cos(qx)| = \sup_{x \geq 0} |\cos(px) - \cos(qx)|$ . An elementary verification shows that  $\lambda > \frac{8}{3\sqrt{3}}$  if  $q \neq 3p$ , and  $\lambda = \frac{8}{3\sqrt{3}}$  if  $q = 3p$ , see for example [10]. Now let  $\mu < \lambda$ , and let  $\eta < \delta$  be two real numbers such that  $|\cos(px) - \cos(qx)| > \mu$  for  $\eta \leq x \leq \delta$ . Since  $\{e^{ian}\}_{n \geq 1} = \{e^{\frac{2n\pi i}{u}}\}_{1 \leq n \leq u}$ , we see that  $\sup_{n \geq 1} |\cos(npa) - \cos(nqa)| > \mu$  if  $\frac{2\pi}{u} < \delta - \eta$ , and the lemma follows.  $\square$

**Lemma 3.4.** *Assume that  $\frac{a}{\pi}$  and  $\frac{b}{\pi}$  are rational, let  $u \geq 1$  be the order of  $a$  and let  $v$  be the order of  $b$ .*

*(i) If  $u \neq v$ ,  $u \neq 3v$ ,  $v \neq 3u$  then  $\sup_{n \geq 1} |\cos(na) - \cos(nb)| \geq 1 + \cos\left(\frac{\pi}{5}\right) > 1.8 > \frac{8}{3\sqrt{3}}$ .*

(ii) If  $u = v$ , and if  $b \notin \pm a + 2\pi\mathbb{Z}$ , then there exists  $w \in \mathbb{Z}$  such that  $2 \leq w \leq \frac{u}{2}$  and  $\gcd(u, w) = 1$  satisfying

$$\sup_{n \geq 1} |\cos(na) - \cos(nb)| = \sup_{n \geq 1} \left| \cos\left(\frac{2n\pi}{u}\right) - \cos\left(\frac{2nw\pi}{u}\right) \right|. \quad (2)$$

Conversely if  $a \in \pi\mathbb{Q}$  has order  $u$ , then for every integer  $w$  such that  $\gcd(w, u) = 1$ , there exists  $b \in \pi\mathbb{Q}$  of order  $u$  satisfying (2).

(iii) If  $v = 3u$ , then there exists an integer  $w$  such that  $1 \leq w \leq \frac{u}{2}$  and  $\gcd(u, w) = 1$  satisfying

$$\sup_{n \geq 1} |\cos(na) - \cos(nb)| = \sup_{n \geq 1} \left| \cos\left(\frac{2n\pi}{3u}\right) - \cos\left(\frac{2nw\pi}{u}\right) \right|. \quad (3)$$

Conversely if  $a \in \pi\mathbb{Q}$  has order  $u$ , then for every integer  $w$  such that  $\gcd(w, u) = 1$  there exists  $b \in \pi\mathbb{Q}$  of order  $3u$  satisfying (3).

(iv) If  $u = 3v$ , then there exists an integer  $w$  such that  $1 \leq w \leq \frac{u}{6}$  and  $\gcd\left(\frac{u}{3}, w\right) = 1$  satisfying

$$\sup_{n \geq 1} |\cos(na) - \cos(nb)| = \sup_{n \geq 1} \left| \cos\left(\frac{2n\pi}{u}\right) - \cos\left(\frac{6nw\pi}{u}\right) \right|. \quad (4)$$

Conversely if the order  $u$  of  $a \in \pi\mathbb{Q}$  is divisible by 3, then for every integer  $w$  such that  $\gcd\left(\frac{u}{3}, w\right) = 1$  there exists  $b \in \pi\mathbb{Q}$  of order  $\frac{u}{3}$  satisfying (4).

Proof : (i) Assume that  $u \neq v$ , say,  $u < v$ , and let  $w \neq 1$  be the order of  $ub$ , which is a divisor of  $v$ . We have  $ub = \frac{2\pi\alpha}{w}$ , with  $\gcd(\alpha, w) = 1$ , and there exists  $\gamma \geq 1$  such that  $\alpha\gamma - 1 \in w\mathbb{Z}$ . We obtain

$$\sup_{n \geq 1} |\cos(na) - \cos(nb)| \geq \sup_{n \geq 1} |\cos(nu\gamma\alpha) - \cos(nu\gamma b)| = \sup_{1 \leq n \leq w} 1 - \cos\left(\frac{2n\pi}{w}\right).$$

If  $w$  is even, then  $\sup_{n \geq 1} |\cos(na) - \cos(nb)| = 2$ . If  $w$  is odd, set  $s = \frac{w-1}{2}$ . We obtain

$$\sup_{n \geq 1} |\cos(na) - \cos(nb)| \geq 1 - \cos\left(\frac{2s\pi}{w}\right) = 1 + \cos\left(\frac{\pi}{w}\right).$$

If  $w \geq 5$ , we obtain

$$\sup_{n \geq 1} |\cos(na) - \cos(nb)| \geq 1 + \cos\left(\frac{\pi}{5}\right) > 1.8 > \frac{8}{3\sqrt{3}}.$$

If  $w = 3$ , let  $d = \gcd(u, v)$ , and set  $r = \frac{u}{d}$ . Then  $w = 3 = \frac{v}{d} > r$ . So either  $r = 1$  or  $r = 2$ .

If  $r = 2$ , we have  $u = 2d, v = 3d, a = \frac{2p\pi}{2d} = \frac{p\pi}{d}$  with  $p$  odd,  $b = \frac{2q\pi}{3d}$  with  $\gcd(q, 3d) = 1$ , and we obtain

$$\begin{aligned} \sup_{n \geq 1} |\cos(na) - \cos(nb)| &= |\cos(3da) - \cos(3db)| \\ &\geq |\cos(3p\pi) - \cos(2q\pi)| = 2. \end{aligned}$$

If  $r = 1$  then  $u = d$  and  $v = 3d = u$ .

We thus see that if  $v > u$  and  $v \neq 3u$ , then  $\sup_{n \geq 1} |\cos(na) - \cos(nb)| \geq 1 + \cos\left(\frac{\pi}{5}\right) > 1.8 > \frac{3}{\sqrt{3}}$ , which proves (i).

(ii) Assume that  $u = v$ , and that  $b \notin \pm a + 2\pi\mathbb{Z}$ . There exists  $\alpha, \beta \in \{1, \dots, u-1\}$ , with  $\alpha \neq \beta, \alpha \neq u - \beta$  such that  $a \in \pm \frac{2\alpha\pi}{u} + 2\pi\mathbb{Z}$  and  $b \in \pm \frac{2\beta\pi}{u} + 2\pi\mathbb{Z}$ , and  $\gcd(\alpha, u) = \gcd(\beta, u) = 1$ . It follows from Bezout's identity that there exists  $\gamma \in \mathbb{Z}$  such that  $\alpha\gamma - 1 \in u\mathbb{Z}$ . If  $\beta\gamma \pm 1 \in u\mathbb{Z}$  then we would have  $\alpha\beta\gamma \pm \alpha \in \alpha u\mathbb{Z} \subset u\mathbb{Z}$ , and  $\beta \pm \alpha \in u\mathbb{Z}$ , which is impossible. Hence  $\gamma\beta - w \in u\mathbb{Z}$  for some  $w \in \{2, \dots, u-2\}$ ,  $\gcd(w, u) = 1$  since  $\gcd(\gamma, u) = \gcd(\beta, u) = 1$ , and we have

$$\begin{aligned} \sup_{n \geq 1} |\cos(na) - \cos(nb)| &\geq \sup_{n \geq 1} |\cos(n\gamma a) - \cos(n\gamma b)| \\ &= \sup_{n \geq 1} \left| \cos\left(\frac{2n\pi}{u}\right) - \cos\left(\frac{2nw\pi}{u}\right) \right| \geq \sup_{n \geq 1} \left| \cos\left(\frac{2n\alpha\pi}{u}\right) - \cos\left(\frac{2n\alpha w\pi}{u}\right) \right| \\ &= \sup_{n \geq 1} \left| \cos\left(\frac{2n\alpha\pi}{u}\right) - \cos\left(\frac{2n\beta\pi}{u}\right) \right| = \sup_{n \geq 1} |\cos(na) - \cos(nb)|. \end{aligned}$$

By replacing  $w$  by  $u - w$  if necessary, we can assume that  $2 \leq w \leq \frac{u}{2}$ .

Now let  $w \in \mathbb{Z}$  such that  $\gcd(u, w) = 1$ . We have  $a = \frac{2\alpha\pi}{u}$ , with  $\gcd(\alpha, u) = 1$ . The same argument as above shows that we have

$$\sup_{n \geq 1} \left| \cos\left(\frac{2n\pi}{u}\right) - \cos\left(\frac{2nw\pi}{u}\right) \right| = \sup_{n \geq 1} |\cos(na) - \cos(nb)|,$$

with  $b = \frac{2w\alpha\pi}{u}$ , which has order  $u$ .

(iii) Now assume that  $v = 3u$ . There exists  $\alpha \in \{1, \dots, u-1\}$  and  $\beta \in \{1, \dots, 3u-1\}$  such that  $a \in \pm \frac{2\alpha\pi}{u} + 2\pi\mathbb{Z}$  and  $b \in \pm \frac{2\beta\pi}{3u} + 2\pi\mathbb{Z}$ , and  $\gcd(\alpha, u) = \gcd(\beta, 3u) = 1$ . Let  $\gamma \in \mathbb{Z}$  such that  $\beta\gamma - 1 \in 3u\mathbb{Z}$ . Then  $\gcd(\gamma, 3u) = 1$ , and a fortiori  $\gcd(\gamma, u) = 1$ . There exists  $w \in \mathbb{Z}$  such that  $\alpha\gamma \in \pm w + u\mathbb{Z}$ , and we see as above that we have

$$\begin{aligned} \sup_{n \geq 1} |\cos(na) - \cos(nb)| &= \sup_{n \geq 1} \left| \cos\left(\frac{2n\alpha\pi}{u}\right) - \cos\left(\frac{2n\beta\pi}{3u}\right) \right| \\ &= \sup_{n \geq 1} \left| \cos\left(\frac{2n\alpha\gamma\pi}{u}\right) - \cos\left(\frac{2n\beta\gamma\pi}{3u}\right) \right| = \sup_{n \geq 1} \left| \cos\left(\frac{2nw\pi}{u}\right) - \cos\left(\frac{2n\pi}{3u}\right) \right|. \end{aligned}$$

Conversely let  $a = \frac{2\alpha\pi}{u} \in \pi\mathbb{Q}$  have order  $u$ , and let  $w \in \mathbb{Z}$  be such that  $\gcd(u, w) = 1$ . If  $\alpha$  is not divisible by 3, then  $\gcd(\alpha, 3u) = 1$ . If  $\alpha$  is divisible by 3, then  $u$  is not divisible by 3, and so  $\alpha + u \in \alpha + u\mathbb{Z}$  is not divisible by 3. So we can assume without loss of generality that  $\alpha$  is not divisible by 3, and there exists  $\beta \geq 1$  such that  $\alpha\beta - 1 \in 3u\pi\mathbb{Z}$ . Similarly we can assume without loss of generality that  $w$  is not divisible by 3, and there exists  $\gamma \geq 1$  such that  $w\gamma - 1 \in 3u\pi\mathbb{Z}$ . Set  $b = \frac{2\alpha\gamma\pi}{3u}$ . Then  $b$  has order  $3u$ , and we see as above that we have

$$\begin{aligned} \sup_{n \geq 1} \left| \cos\left(\frac{2nw\pi}{u}\right) - \cos\left(\frac{2n\pi}{3u}\right) \right| &\geq \sup_{n \geq 1} \left| \cos\left(\frac{2n\alpha\gamma w\pi}{u}\right) - \cos\left(\frac{2n\alpha\gamma\pi}{3u}\right) \right| \\ &= \sup_{n \geq 1} |\cos(na) - \cos(nb)| \geq \sup_{n \geq 1} \left| \cos\left(\frac{2n\alpha\gamma w\beta w\pi}{u}\right) - \cos\left(\frac{2n\alpha\gamma\beta w\pi}{3u}\right) \right| \\ &= \sup_{n \geq 1} \left| \cos\left(\frac{2nw\pi}{u}\right) - \cos\left(\frac{2n\pi}{3u}\right) \right|, \end{aligned}$$

which concludes the proof of (iii).

(iv) Clearly, the first assertion of (iv) is a reformulation of the first assertion of (iii). Now assume that the order  $u$  of  $a \in \pi\mathbb{Q}$  is divisible by 3, set  $v = \frac{u}{3}$ , write  $a = \frac{2\alpha\pi}{u}$ , and let  $w \in \mathbb{Z}$  such that  $\gcd(w, v) = 1$ . We see as above that we can assume without loss of generality that  $\gcd(u, w) = 1$ .

Since  $\gcd(\alpha, u) = 1$ , we have a fortiori  $\gcd(\alpha, v) = 1$ , so that  $\gcd(\alpha w, v) = 1$ , so that  $b := \frac{6\alpha w}{u}$  has order  $v$  and we see as above that  $a, b, u$  and  $w$  satisfy (4).  $\square$

In order to use lemma 3.4, we introduce the following notions.

**Definition 3.5.** Let  $u \geq 2$ , and denote by  $\Delta(u)$  the set of all integers  $s$  satisfying  $1 \leq s \leq \frac{u}{2}$ ,  $\gcd(u, s) = 1$ , and let  $\Delta_1(u) = \Delta(u) \setminus \{1\}$ . We set

$$\begin{aligned} \sigma(u) &= \inf_{w \in \Delta(u)} \left[ \sup_{n \geq 1} \left| \cos\left(\frac{2\pi}{3u}\right) - \cos\left(\frac{2w\pi}{u}\right) \right| \right], \\ \theta(u) &= \inf_{w \in \Delta_1(u)} \left[ \sup_{n \geq 1} \left| \cos\left(\frac{2\pi}{u}\right) - \cos\left(\frac{2w\pi}{u}\right) \right| \right]. \end{aligned}$$

with the convention  $\theta(u) = 2$  if  $\Delta_1(u) = \emptyset$ .

Notice that  $\Delta_1(u) = \emptyset$  if  $u = 2, 3, 4$  or  $6$ , and that  $\Delta_1(u) \neq \emptyset$  otherwise since as we observed above  $H(n) = \text{card}((\mathbb{Z}/n\mathbb{Z})^\times) \geq 3$  if  $n \notin \{1, 2, 3, 4, 6\}$ .

We obtain the following corollary, which shows in particular that the value of  $k(a)$  depends only on the order of  $a$ .

**Corollary 3.6.** Let  $a \in \pi\mathbb{Q}$ , and let  $u \geq 1$  be the order of  $a$ .

- (i) If  $u$  is not divisible by 3, then  $k(a) = \inf(\sigma(u), \theta(u))$ .
- (ii) If  $u$  is divisible by 3, then  $k(a) = \inf(\sigma(\frac{u}{3}), \sigma(u), \theta(u))$ .

Proof : Set

- $\Lambda_1(a) = \{b \in \pi\mathbb{Q} \mid b \notin \pm a + 2\pi\mathbb{Z}, \text{ord}(b) = \text{ord}(a)\},$
- $\Lambda_2(a) = \{b \in \pi\mathbb{Q} \mid \text{ord}(b) = 3\text{ord}(a)\},$
- $\Lambda_3(a) = \{b \in \pi\mathbb{Q} \mid 3\text{ord}(b) = \text{ord}(a)\},$
- $\Lambda_4(a) = \{b \in \pi\mathbb{Q} \mid \text{ord}(b) \neq \text{ord}(a) \neq 3\text{ord}(b)\},$

and for  $1 \leq i \leq 4$ , set

$$\lambda_i(a) = \inf_{b \in \Lambda_i(a)} \sup_{n \geq 1} |\cos(na) - \cos(nb)|,$$

with the convention  $\lambda_i(a) = 2$  if  $\Lambda_i(a) = \emptyset$ .

Since  $b \notin \pm a + 2\pi\mathbb{Z}$  if  $\text{ord}(b) \neq \text{ord}(a)$ , we have  $\lambda_2(a) \leq \frac{8}{3\sqrt{3}}$ , and it follows from lemma 3.4(i) that we have

$$k(a) = \inf_{1 \leq i \leq 4} \lambda_i(a) = \inf_{1 \leq i \leq 3} \lambda_i(a),$$

and it follows from lemma 3.4 (ii), (iii) and (iv) that  $\lambda_1(a) = \theta(u)$  if  $\Delta_1(u) \neq \emptyset$ , that  $\lambda_2(a) = \sigma(u)$ , and that  $\lambda_3(a) = \sigma\left(\frac{u}{3}\right)$  if  $u$  is divisible by 3.  $\square$

We have the following result.

**Theorem 3.7.** *Let  $m < \frac{8}{3\sqrt{3}}$ . Then the set  $\Omega(m) := \{a \in [0, \pi] : k(a) \leq m\}$  is finite.*

Proof : It follows from lemma 3.3 applied to  $\frac{2\pi}{u}$  and  $\frac{6\pi}{u}$  that there exists  $u_0 \geq 1$  such that we have, for  $u \geq u_0$ ,

$$(i) \sup_{n \geq 1} \left| \cos\left(\frac{2n\pi}{u}\right) - \cos\left(\frac{2wn\pi}{u}\right) \right| > m \text{ if } 2 \leq w \leq \inf\left(\frac{u}{2}, 6\right),$$

$$(ii) \sup_{n \geq 1} \left| \cos\left(\frac{6n\pi}{u}\right) - \cos\left(\frac{2(3w+1)n\pi}{u}\right) \right| > m \text{ if } 0 \leq w \leq 6,$$

$$(iii) \sup_{n \geq 1} \left| \cos\left(\frac{6n\pi}{u}\right) - \cos\left(\frac{2(3w+2)n\pi}{u}\right) \right| > m \text{ if } 0 \leq w \leq 6.$$

Let  $u \geq u_0$ , and let  $w$  be an integer such that  $2 \leq w \leq \frac{u}{2}$ . If  $\frac{2w\pi}{u} \leq \pi/2$ , or if  $\frac{2w\pi}{u} \geq \frac{5\pi}{6}$ , it follows from lemma 3.2 and property (i) that we have

$$\sup_{n \geq 1} \left| \cos\left(\frac{2n\pi}{u}\right) - \cos\left(\frac{2wn\pi}{u}\right) \right| > m.$$

Now assume that  $\frac{\pi}{2} \leq \frac{2w\pi}{u} \leq \frac{5\pi}{6}$ . If  $|w - \frac{u}{3}| \geq 7$ , it follows from lemma 3.2 that we have

$$\sup_{n \geq 1} \left| \cos\left(\frac{2n\pi}{u}\right) - \cos\left(\frac{2wn\pi}{u}\right) \right| > 1.55 > m.$$

If  $|w - \frac{u}{3}| < 7$ , set  $r = |3w - u|$ . Then  $0 \leq r \leq 20$ , and we have

$$\sup_{n \geq 1} \left| \cos\left(\frac{2n\pi}{u}\right) - \cos\left(\frac{2wn\pi}{u}\right) \right| \geq \sup_{n \geq 1} \left| \cos\left(\frac{6n\pi}{u}\right) - \cos\left(\frac{2nr\pi}{u}\right) \right|.$$

If  $u$  is not divisible by 3, then either  $r = 3s + 1$  or  $r = 3s + 2$ , with  $0 \leq w \leq 6$ , and it follows from (ii) and (iii) that we have

$$\sup_{n \geq 1} \left| \cos\left(\frac{2n\pi}{u}\right) - \cos\left(\frac{2wn\pi}{u}\right) \right| > m.$$

If  $u$  is divisible by 3 then  $r$  is also divisible by 3. Set  $v = \frac{u}{3}$  and  $s = \frac{r}{3}$ . Then  $0 \leq s \leq 6$ , and we have

$$\sup_{n \geq 1} \left| \cos\left(\frac{2n\pi}{u}\right) - \cos\left(\frac{2wn\pi}{u}\right) \right| \geq \sup_{n \geq 1} \left| \cos\left(\frac{2n\pi}{v}\right) - \cos\left(\frac{2sn\pi}{v}\right) \right|.$$

If  $s \in \{2, 3, 4, 5, 6\}$  it follows from (i) that we have, if  $u \geq 3u_0$ ,

$$\sup_{n \geq 1} \left| \cos\left(\frac{2n\pi}{v}\right) - \cos\left(\frac{2sn\pi}{v}\right) \right| > m.$$

Now assume that  $s = 0$ . If  $u \geq 15$ , then  $v \geq 5$ , and we have

$$\sup_{n \geq 1} \left| \cos\left(\frac{2n\pi}{v}\right) - \cos\left(\frac{2sn\pi}{u}\right) \right| = \sup_{n \geq 1} \left| \cos\left(\frac{2n\pi}{v}\right) - 1 \right| \geq 1 + \cos\left(\frac{\pi}{5}\right) > 1.8 > m.$$

Now assume that  $s = 1$ . We have, with  $\epsilon = \pm 1$ ,

$$\begin{aligned} \sup_{n \geq 1} \left| \cos\left(\frac{2n\pi}{u}\right) - \cos\left(\frac{2wn\pi}{u}\right) \right| &= \sup_{n \geq 1} \left| \cos\left(\frac{2n\pi}{3v}\right) - \cos\left(\frac{2n\pi}{3v} + \frac{2n\epsilon\pi}{3}\right) \right| \\ &\geq \sup_{n \geq 1} \left| \cos\left(\frac{2(3n+1)\pi}{3v}\right) - \cos\left(\frac{2(3n+1)\pi}{3v} + \frac{2\epsilon\pi}{3}\right) \right| = \sqrt{3} \left| \sin\left(\frac{2n\pi}{v} + \frac{2\pi}{3v} + \frac{\epsilon\pi}{3}\right) \right|. \end{aligned}$$

There exists  $p \geq 1$  and  $q \in \mathbb{Z}$  such that  $\frac{\pi}{2} - \frac{\pi}{v} \leq \frac{2p\pi}{v} + \frac{2\pi}{3v} + \frac{\epsilon\pi}{3} + 2q\pi \leq \frac{\pi}{2} + \frac{\pi}{v}$ , and we obtain, for  $u \geq 21$ ,  $w = v \pm 1$ ,

$$\sup_{n \geq 1} \left| \cos\left(\frac{2n\pi}{u}\right) - \cos\left(\frac{2wn\pi}{u}\right) \right| \geq \sqrt{3} \cos\left(\frac{\pi}{v}\right) \geq \sqrt{3} \cos\left(\frac{\pi}{7}\right) \geq 1.56 > m.$$

We thus see that if  $u \geq u_0$  is not divisible par 3, or if  $u \geq \max(21, 3u_0)$  is divisible by 3, we have, for  $2 \leq w \leq \frac{u}{2}$ ,



$$\sup_{n \geq 1} \left| \cos\left(\frac{2n\pi}{u}\right) - \cos\left(\frac{2wn\pi}{u}\right) \right| \geq m.$$

It follows then from corollary 3.6 that if the order  $u$  of  $a \in [0, 2\pi]$  satisfies  $u \geq \max(21, 3u_0)$ , we have  $k(a) > m$ .

□

We now want to identify the real numbers  $a$  for which  $k(a) \leq 1.5$ .

If  $a \in \pi\mathbb{Q}$  has order 1, 2 or 4, then  $\sup_{n \geq 1} |\cos(an) - \cos(3an)| = 0$ . We also have the following elementary facts.

**Lemma 3.8.** *Let  $a \in \pi\mathbb{Q}$ , and let  $u \notin \{1, 2, 4\}$  be the order of  $a$ .*

1. *If  $u \notin \{3, 5, 6, 8, 9, 10, 11, 12, 15, 16, 18, 22, 24, 30\}$  then*

$$\sup_{n \geq 1} |\cos(an) - \cos(3an)| > 1.5.$$

2. *If  $u \in \{3, 6, 9, 12, 15, 18, 24, 30\}$ , then*

$$\sup_{n \geq 1} |\cos(an) - \cos(3an)| = 1.5.$$

3. *If  $u \in \{5, 10\}$ , then*

$$\sup_{n \geq 1} |\cos(an) - \cos(3an)| = \frac{\sqrt{5}}{2}.$$

4. *If  $u \in \{8, 16\}$ , then*

$$\sup_{n \geq 1} |\cos(an) - \cos(3an)| = \sqrt{2}.$$

5. *If  $u \in \{11, 22\}$ , then*

$$\sup_{n \geq 1} |\cos(an) - \cos(3an)| = -\cos\left(\frac{8\pi}{11}\right) + \cos\left(\frac{24\pi}{11}\right) = \cos\left(\frac{2\pi}{11}\right) + \cos\left(\frac{3\pi}{11}\right) \approx 1.4961.$$

Proof : We have  $\{e^{ian}\}_{n \geq 1} = \{e^{\frac{2in\pi}{u}}\}_{1 \leq n \leq u}$ , and so we have

$$\begin{aligned} \sup_{n \geq 1} |\cos(an) - \cos(3an)| &= \sup_{n \geq 1} \left| \cos\left(\frac{2n\pi}{u}\right) - \cos\left(\frac{6n\pi}{u}\right) \right| \\ &= \sup_{1 \leq n \leq u} \left| \cos\left(\frac{2n\pi}{u}\right) - \cos\left(\frac{6n\pi}{u}\right) \right|, \end{aligned}$$

and the value of  $\sup_{n \geq 1} |\cos(an) - \cos(3an)|$  depends only on the order  $u$  of  $a$ .

The function  $x \rightarrow \cos(x) - \cos(3x)$  is increasing on  $\left[0, \arccos\left(\frac{1}{\sqrt{3}}\right)\right]$  and decreasing on  $\left[\arccos\left(\frac{1}{\sqrt{3}}\right), -\arccos\left(\frac{1}{\sqrt{3}}\right)\right]$ , and  $0.275\pi < \arccos\left(\frac{1}{\sqrt{3}}\right) < 0.333\pi$ . Since  $\cos(x) - \cos(3x) > 1.5$  if  $x = 0.275\pi$  or if  $x = 0.333\pi$ , there exists a closed interval  $I$  of length  $0.058\pi$  on which  $\cos(x) - \cos(3x) > 1.5$ . So if  $u \geq 35 > \frac{2}{0.058}$ , there exists  $n \geq 1$  such that  $\frac{2n\pi}{u} \in I$ , and we have

$$\sup_{n \geq 1} |\cos(an) - \cos(3an)| > 1.5 \quad \forall n \geq 35.$$

The other properties follow from computations of  $\sup_{1 \leq n \leq u} \left| \cos\left(\frac{2n\pi}{u}\right) - \cos\left(\frac{6n\pi}{u}\right) \right|$  for  $3 \leq u \leq 34$  which are left to the reader.  $\square$

We now wish to obtain similar estimates for  $\sup_{n \geq 1} \left| \cos\left(\frac{2\pi}{n}\right) - \cos\left(\frac{2s\pi}{n}\right) \right|$  for  $s \in \{2, 4, 5, 6\}$ . Set  $f_s(x) = \cos(x) - \cos(sx)$ ,  $\theta_s = \sup_{x \geq 0} |f(s)|$ ,  $\delta_s = \sup_{x \geq 0} |f''(s)|$ . We have  $\theta_s = 2$  if  $s$  is even, and a computer verification shows that  $\theta_s > 1.8$  for  $s = 5$ . It follows from the Taylor-Lagrange inequality that if  $f_s$  attains its maximum at  $\alpha_s$ , then we have,

$$|f_s(x) - \theta_s| \leq \frac{\delta_s}{2} |(x - \alpha_s)^2|, |f_s(x)| \geq \theta_s - \frac{\delta_s}{2} |(x - \alpha_s)^2|,$$

and so  $|f_s(x)| > 1.5$  if  $|(x - \alpha_s)^2| \leq \frac{2\theta_s - 3}{\delta_s}$ . So if  $l_s < \sqrt{\frac{2\theta_s - 3}{\delta_s}}$  there exists a closed interval of length  $2l_s$  on which  $|f_s(x)| > 1.5$ . Let  $u_s \geq \frac{\pi}{l_s}$  be an integer. We obtain

$$\sup_{n \geq 1} \left| \cos\left(\frac{2n\pi}{u}\right) - \cos\left(\frac{2sn\pi}{u}\right) \right| > 1.5 \quad \forall u \geq u_s. \quad (5)$$

Values for  $u_s$  are given by the following table.

$s$	$\theta_s$	$\delta_s$	$l_s$	$u_s$
2	2	$\leq 5$	0.4472	8
4	2	$\leq 17$	0.2425	13
5	$> 1.8$	$\leq 26$	0.1519	21
6	2	$\leq 37$	0.1644	20

We obtain the following result

**Lemma 3.9.** *Let  $u \geq 4$  be an integer, and let  $s \leq \frac{u}{4}$  be a nonnegative integer. If  $s \neq 3$ , then we have*

$$\sup_{n \geq 1} \left| \cos\left(\frac{2n\pi}{u}\right) - \cos\left(\frac{2sn\pi}{u}\right) \right| > 1.5$$

Proof : If  $s = 0$ , then

$$\sup_{n \geq 1} \left| \cos\left(\frac{2n\pi}{u}\right) - \cos\left(\frac{2ns\pi}{u}\right) \right| = \sup_{n \geq 1} \left| \cos\left(\frac{2n\pi}{u}\right) - 1 \right| > 1.8.$$

If  $s \geq 7$ , the result follows from lemma 3.2 (i). If  $s \in \{2, 4, 6\}$ , the result follows from the table since  $u \geq 4s$ . If  $s = 5$ , the result also follows from the table for  $u \geq 21$ , and a direct computation shows that we have

$$\sup_{n \geq 1} \left| \cos\left(\frac{2n\pi}{20}\right) - \cos\left(\frac{10n\pi}{20}\right) \right| = \sup_{1 \leq n \leq 20} \left| \cos\left(\frac{n\pi}{10}\right) - \cos\left(\frac{n\pi}{2}\right) \right| = 1 + \cos\left(\frac{\pi}{5}\right) > 1.80.$$

□

Now set  $g_s(x) = \cos(3x) - \cos(sx)$ ,  $\theta_s = \sup_{x \geq 0} |g(s)|$ ,  $\delta_s = \sup_{x \geq 0} |g''(s)|$ . We have  $\theta_s = 2$  if  $s$  is even, and a computer verification shows that  $\theta_s > 1.85$  for  $s = 5$ ,  $\theta_s > 1.91$  for  $s = 7, s = 11$ ,  $\theta_s > 1.97$  for  $s = 13, s = 17$ ,  $\theta_s > 1.96$  for  $s = 19$ . We see as above that if  $l_s < \sqrt{\frac{2\theta_s - 3}{\delta_s}}$ , and if  $u_s \geq \frac{\pi}{l_s}$  is an integer, we have

$$\sup_{n \geq 1} \left| \cos\left(\frac{2sn\pi}{u}\right) - \cos\left(\frac{6n\pi}{u}\right) \right| > 1.5 \quad \forall u \geq u_s. \quad (6)$$

We have the following table.

$s$	$\theta_s$	$\delta_s$	$l_s$	$u_s$
2	2	$\leq 13$	0.2774	12
4	2	$\leq 23$	0.2085	16
5	$> 1.85$	$\leq 34$	0.1435	22
7	$> 1.91$	$\leq 58$	0.1189	27
8	2	$\leq 73$	0.1170	27
10	2	$\leq 109$	0.0958	33
11	$> 1.91$	$\leq 130$	0.0794	40
13	$> 1.97$	$\leq 178$	0.0727	44
14	2	$\leq 205$	0.0698	45
16	2	$\leq 275$	0.0603	53
17	$> 1.97$	$\leq 298$	0.0562	56
19	$> 1.96$	$\leq 390$	0.0486	65
20	2	$\leq 409$	0.0494	64

We will be interested here to the case where  $u$  is not divisible by 3 and where  $\frac{2s\pi}{u} \leq \frac{\pi}{2}$ , which means that  $u \geq 4s$ . So we are left with  $s = 2$ ,  $u = 8, 10$  or  $11$ , and with  $s = 5$ ,  $u = 20$ . We obtain, by direct computations

$$\sup_{n \geq 1} \left| \cos\left(\frac{4n\pi}{8}\right) - \cos\left(\frac{6n\pi}{8}\right) \right| = \sup_{n \geq 1} \left| \cos\left(\frac{n\pi}{2}\right) - \cos\left(\frac{3n\pi}{4}\right) \right| = 2.$$

$$\sup_{n \geq 1} \left| \cos\left(\frac{4n\pi}{10}\right) - \cos\left(\frac{6n\pi}{10}\right) \right| = \sup_{n \geq 1} \left| \cos\left(\frac{2n\pi}{5}\right) - \cos\left(\frac{3n\pi}{5}\right) \right| = 2.$$

$$\sup_{n \geq 1} \left| \cos\left(\frac{4n\pi}{11}\right) - \cos\left(\frac{6n\pi}{11}\right) \right| = \cos\left(\frac{20\pi}{11}\right) - \cos\left(\frac{30\pi}{11}\right) = \cos\left(\frac{2\pi}{11}\right) + \cos\left(\frac{3\pi}{11}\right) \approx 1.4961.$$

$$\sup_{n \geq 1} \left| \cos\left(\frac{10n\pi}{20}\right) - \cos\left(\frac{6n\pi}{20}\right) \right| = \sup_{n \geq 1} \left| \cos\left(\frac{n\pi}{2}\right) - \cos\left(\frac{3n\pi}{10}\right) \right| > 1.80.$$

We obtain the following lemma.

**Lemma 3.10.** *Let  $u, s$  be positive integers satisfying  $u \geq 4$ ,  $\frac{u}{4} \leq s \leq \frac{5u}{12}$ , with  $s \geq 2$ , so that  $u \geq 5$ . We have*

$$\sup_{n \geq 1} \left| \cos\left(\frac{2n\pi}{u}\right) - \cos\left(\frac{2sn\pi}{u}\right) \right| \begin{cases} = \cos\left(\frac{\pi}{5}\right) + \cos\left(\frac{2\pi}{5}\right) & \text{if } u = 5, s = 2, \text{ or if } u = 10, s = 3, \\ = \sqrt{2} & \text{if } u = 8, s = 3, \text{ or if } u = 16, s = 5, \\ = \cos\left(\frac{2\pi}{11}\right) + \cos\left(\frac{3\pi}{11}\right) & \text{if } u = 11, s = 4 \text{ or if } u = 22, s = 7, \\ = 1.5 & \text{if } u = 12, s = 3, \\ > 1.5 & \text{otherwise.} \end{cases}$$

Proof : Set  $r = |3s - u|$ . Since  $\frac{2\pi}{3} - \frac{\pi}{2} = \frac{5\pi}{6} - \frac{2\pi}{3} = \frac{\pi}{6}$ , we have  $0 \leq \frac{2\pi r}{u} \leq \frac{\pi}{2}$ . If  $r \geq 21$ , it follows from lemma 3.1(i) that  $\sup_{n \geq 1} \left| \cos\left(\frac{2n\pi}{u}\right) - \cos\left(\frac{2sn\pi}{u}\right) \right| > 1.5$ .

If  $u$  is not divisible by 3, then  $r$  is not divisible by 3 either, and it follows from the discussion above that if  $r \neq 1$  and  $r \neq 2$ , we have

$$\sup_{n \geq 1} \left| \cos\left(\frac{2n\pi}{u}\right) - \cos\left(\frac{2sn\pi}{u}\right) \right| \geq \sup_{n \geq 1} \left| \cos\left(\frac{6n\pi}{u}\right) - \cos\left(\frac{2rn\pi}{u}\right) \right| > 1.5.$$

The condition  $r = 2$  gives  $|s - \frac{u}{3}| = \frac{2}{3}$ . We saw above that in this situation  $\sup_{n \geq 1} \left| \cos\left(\frac{2n\pi}{u}\right) - \cos\left(\frac{2sn\pi}{u}\right) \right| > 1.5$  unless  $u = 11$ , which gives  $s = 3$ , and we have

$$\sup_{n \geq 1} \left| \cos\left(\frac{2n\pi}{11}\right) - \cos\left(\frac{6n\pi}{11}\right) \right| = \sup_{1 \leq n \leq 11} \left| \cos\left(\frac{6n\pi}{11}\right) - \cos\left(\frac{4n\pi}{11}\right) \right|$$

$$= \left| \cos\left(\frac{30\pi}{11}\right) + \cos\left(\frac{20\pi}{11}\right) \right| = \cos\left(\frac{2\pi}{11}\right) + \cos\left(\frac{3\pi}{11}\right) \approx 1.4961.$$

The condition  $r = 1$  gives  $|s - \frac{u}{3}| = \frac{1}{3}$ , which gives  $s = \frac{u-1}{3}$  if  $u \equiv 1 \pmod{3}$ , and  $s = \frac{u+1}{3}$  if  $u \equiv 2 \pmod{3}$ . In this situation we have

$$\sup_{n \geq 1} \left| \cos\left(\frac{2n\pi}{u}\right) - \cos\left(\frac{2sn\pi}{u}\right) \right| \geq \sup_{n \geq 1} \left| \cos\left(\frac{2n\pi}{u}\right) - \cos\left(\frac{6n\pi}{u}\right) \right|.$$

Since we must have  $|s - \frac{u}{3}| = \frac{1}{3}$ , it follows from lemma 3.8 that if  $n \notin \{5, 8, 10, 11, 16, 22\}$ , or if  $u = 5, s \neq 2$ , or if  $u = 8, s \neq 3$ , or if  $u = 10, s \neq 3$ , or if  $u = 11, s \neq 4$ , or if  $u = 16, s \neq 5$ , or if  $u = 22, s \neq 7$  we have

$$\sup_{n \geq 1} \left| \cos\left(\frac{2n\pi}{u}\right) - \cos\left(\frac{2sn\pi}{u}\right) \right| > 1.5.$$

A direct computation shows the that we have

$$\begin{aligned} \sup_{n \geq 1} \left| \cos\left(\frac{2n\pi}{u}\right) - \cos\left(\frac{2sn\pi}{u}\right) \right| &= \sup_{1 \leq n \leq u} \left| \cos\left(\frac{2n\pi}{u}\right) - \cos\left(\frac{2sn\pi}{u}\right) \right| \\ &= \begin{cases} \cos\left(\frac{\pi}{5}\right) + \cos\left(\frac{2\pi}{5}\right) & \text{if } u = 5, s = 2, \text{ or if } u = 10, s = 3, \\ \sqrt{2} & \text{if } u = 8, s = 3, \text{ or if } u = 16, s = 5, \\ \cos\left(\frac{2\pi}{11}\right) + \cos\left(\frac{3\pi}{11}\right) & \text{if } u = 11, s = 4 \text{ or if } u = 22, s = 7. \end{cases} \end{aligned}$$

We now consider the case where  $u = 3v$  is divisible by 3. Then  $r$  is also divisible by 3. If  $r = 0$ , and if  $u \neq 9$ , then we have

$$\sup_{n \geq 1} \left| \cos\left(\frac{2n\pi}{u}\right) - \cos\left(\frac{2sn\pi}{u}\right) \right| \geq \sup_{n \geq 1} \left| \cos\left(\frac{2n\pi}{v}\right) - 1 \right| > 1.8.$$

If  $u = 9$ , then  $s = 3$ , and we have

$$\sup_{n \geq 1} \left| \cos\left(\frac{2n\pi}{u}\right) - \cos\left(\frac{2sn\pi}{u}\right) \right| = \sup_{1 \leq n \leq 9} \left| \cos\left(\frac{2n\pi}{9}\right) - \cos\left(\frac{2n\pi}{3}\right) \right| = 1.5.$$

Now assume that  $r = 3$ , which means that  $s = v + \epsilon$ , with  $\epsilon = \pm 1$ . We have

$$\begin{aligned} \sup_{n \geq 1} \left| \cos\left(\frac{2n\pi}{u}\right) - \cos\left(\frac{2sn\pi}{u}\right) \right| &= \sup_{1 \leq n \leq 3v} \left| \cos\left(\frac{2n\pi}{3v}\right) - \cos\left(\frac{2n\pi}{3} + \frac{2\epsilon n\pi}{3v}\right) \right| \\ &= 2 \sup_{1 \leq n \leq 3v} \left| \sin\left(\frac{n\pi}{3} + \frac{(1+\epsilon)n\pi}{3v}\right) \right| \left| \sin\left(-\frac{n\pi}{3} + \frac{(1-\epsilon)n\pi}{3v}\right) \right| \end{aligned}$$

$$\begin{aligned}
&= \sup_{1 \leq n \leq 3v} \left| \sin\left(\frac{n\pi}{3}\right) \right| \left| \sin\left(\frac{n\pi}{3} + \frac{2n\pi}{3v}\right) \right| \\
&\geq \sqrt{3} \sup_{0 \leq n \leq v} \left| \sin\left(\frac{(3n+1)\pi}{3} + \frac{2(3n+1)\pi}{3v}\right) \right| \\
&= \sqrt{3} \sup_{0 \leq n \leq v} \left| \sin\left(\frac{2n\pi}{v} + \frac{(\nu+2)\pi}{3v}\right) \right|.
\end{aligned}$$

Since  $\sin(x) > \frac{\sqrt{3}}{2}$  for  $\frac{\pi}{3} < x < \frac{2\pi}{3}$ , there exists  $n \in \{1, \dots, v\}$  such that  $\sin\left(\frac{2n\pi}{v} + \frac{(\nu+2)\pi}{3v}\right) > \frac{\sqrt{3}}{2}$  if  $v \geq 7$ , and we obtain

$$\sup_{n \geq 1} \left| \cos\left(\frac{2n\pi}{u}\right) - \cos\left(\frac{2sn\pi}{u}\right) \right| > 1.5 \text{ if } u \geq 21.$$

We are left with the cases where  $u = 6, v = 2, s = 1$  or  $3$ ,  $u = 9, v = 3, s = 2$  or  $4$ ,  $u = 12, v = 4, s = 3$  or  $5$ ,  $u = 15, v = 5, s = 4$  or  $6$ ,  $u = 18, v = 6, s = 5$  or  $7$ . But  $s = 1$  is not relevant, and the condition  $\frac{u}{4} \leq s \leq \frac{5u}{12}$  is not satisfied for  $u = 6, s = 3$  and for  $u = 9, s = 2$  or  $4$ .

Direct computations which are left to the reader show that we have

$$\sup_{n \geq 1} \left| \cos\left(\frac{2n\pi}{u}\right) - \cos\left(\frac{2sn\pi}{u}\right) \right| \begin{cases} > 1.64 \text{ if } u = 15 \text{ and } s = 4, \\ > 1.70 \text{ or if } u = 18 \text{ and } s = 5 \text{ or } s = 7 \\ > 1.72 \text{ if } u = 15 \text{ and } s = 6, \\ > 1.73 \text{ if } u = 12 \text{ and } s = 5. \end{cases}$$

So  $\sup_{n \geq 1} \left| \cos\left(\frac{2n\pi}{u}\right) - \cos\left(\frac{2sn\pi}{u}\right) \right| > 1.5$  if  $\frac{u}{4} \leq s \leq \frac{5u}{12}$  when  $u$  is divisible by 3 and when  $s - \frac{u}{3} \in \{-1, 0, 1\}$ , unless  $u = 12$  and  $s = 3$ . If  $u = 12$  and  $s = 3$ , we have

$$\sup_{n \geq 1} \left| \cos\left(\frac{2n\pi}{u}\right) - \cos\left(\frac{2sn\pi}{u}\right) \right| = \sup_{n \geq 1} \left| \cos\left(\frac{n\pi}{6}\right) - \cos\left(\frac{n\pi}{2}\right) \right| = 1.5.$$

Now assume that  $u = 3v$  is divisible by 3, and that  $2 \leq |s - v| \leq 6$ . Set again  $r = |3s - u|$ , and set  $p = \frac{r}{3}$ , so that  $2 \leq p \leq 6$ . Notice also that  $p \leq \frac{u}{12}$  since  $r \leq \frac{u}{4}$ , so that  $u \geq 24$  and  $v \geq 8$ . We have

$$\begin{aligned}
\sup_{n \geq 1} \left| \cos\left(\frac{2n\pi}{u}\right) - \cos\left(\frac{2sn\pi}{u}\right) \right| &\geq \sup_{n \geq 1} \left| \cos\left(\frac{6n\pi}{u}\right) - \cos\left(\frac{2rn\pi}{u}\right) \right| \\
&= \sup_{n \geq 1} \left| \cos\left(\frac{2n\pi}{v}\right) - \cos\left(\frac{2pn\pi}{u}\right) \right|.
\end{aligned}$$

It follows then from lemma 3.9 that  $\sup_{n \geq 1} \left| \cos\left(\frac{2n\pi}{u}\right) - \cos\left(\frac{2sn\pi}{u}\right) \right| > 1.5$  if  $p \neq 3$ .

If  $p = 3$ , then  $u \geq 36$ , and so  $v \geq 12$ . Since  $s - v = \pm 3$ , it follows from lemma 3.5 that we only have to consider the following cases :

- $u = 36, s = 9$  or  $15$ ,
- $u = 45, s = 12$  or  $18$ ,
- $u = 54, s = 15$  or  $21$ ,
- $u = 72, s = 21$  or  $27$ ,
- $u = 90, s = 27$  or  $33$ .

Direct computations which are left to the reader show that we have

$$\sup_{n \geq 1} \left| \cos\left(\frac{2n\pi}{u}\right) - \cos\left(\frac{2sn\pi}{u}\right) \right| \begin{cases} > 1.93 \text{ if } u = 36 \text{ and } s = 9, \text{ or if } u = 45 \text{ and } s = 12 \text{ or } 18, \\ & \text{or if } u = 72 \text{ and } s = 27, \text{ or if } u = 90 \text{ and } s = 27 \text{ or } 33, \\ > 1.91 \text{ or if } u = 54 \text{ and } s = 15, \\ > 1.87 \text{ or if } u = 72 \text{ and } s = 24, \\ > 1.85 \text{ or if } u = 36 \text{ and } s = 15, \\ > 1.83 \text{ or if } u = 54 \text{ and } s = 21. \end{cases}$$

This concludes the proof of the lemma.  $\square$

**Lemma 3.11.** *Let  $u, s$  be positive integers satisfying  $\frac{5u}{12} \leq s \leq \frac{u}{2}$ , with  $s \geq 2$ , so that  $u \geq 4$ . We have*

$$\sup_{n \geq 1} \left| \cos\left(\frac{2n\pi}{u}\right) - \cos\left(\frac{2sn\pi}{u}\right) \right| = \begin{cases} = 1.5 \text{ if } u = 6 \text{ and } s = 3, \\ > 1.5 \text{ otherwise} \end{cases}$$

Proof : If  $s \geq 4$ , it follows from lemma 3.2(ii) that we have

$$\sup_{n \geq 1} \left| \cos\left(\frac{2n\pi}{u}\right) - \cos\left(\frac{2sn\pi}{u}\right) \right| > 1.57.$$

So we only have to consider the cases  $s = 3, u = 6$  or  $7$ , and  $s = 2, u = 4$ .

A direct computation then shows that we have

$$\sup_{n \geq 1} \left| \cos\left(\frac{2n\pi}{u}\right) - \cos\left(\frac{2sn\pi}{u}\right) \right| \begin{cases} = 2 \text{ if } u = 4 \text{ and } s = 2, \\ = 1.5 \text{ if } u = 6 \text{ and } s = 3, \\ = \cos\left(\frac{2\pi}{7}\right) + \cos\left(\frac{\pi}{7}\right) \approx 1.5245 \text{ if } u = 7 \text{ and } s = 3. \end{cases}$$

$\square$

We consider again the numbers  $\theta(u)$  and  $\sigma(u)$  introduced in definition 3.6.

It follows from lemma 3.8, lemma 3.9, lemma 3.10 and lemma 3.11 that we have the following results.

**Lemma 3.12.** *We have  $\theta(5) = \theta(10) = \cos\left(\frac{\pi}{5}\right) + \cos\left(\frac{2\pi}{5}\right)$ ,  $\theta(8) = \theta(16) = \sqrt{2}$ ,  $\theta(11) = \theta(22) = \cos\left(\frac{2\pi}{11}\right) + \cos\left(\frac{3\pi}{11}\right)$ , and  $\theta(u) > 1.5$  for  $u \geq 4, u \neq 5, u \neq 8, u \neq 10, u \neq 11, u \neq 16, u \neq 22$ .*

**Lemma 3.13.** *We have  $\sigma(u) = 1.5$  if  $u \in \{1, 2, 3, 4, 5, 6, 8, 10\}$ ,  $\sigma(u) > 1.5$  otherwise.*

Hence if  $u$  is divisible by 3, we have  $\sigma\left(\frac{u}{3}\right) = 1.5$  if  $u \in \{3, 6, 9, 12, 15, 18, 24, 30\}$ ,  $\sigma(u) > 1.5$  otherwise. We then deduce from corollary 3.6 a complete description of the set  $\Omega(1.5) = \{a \in [0, \pi] \mid k(a) \leq 1.5\}$ .

**Theorem 3.14.** *Let  $a \in [0, \pi]$ .*

- *If  $a \in \{\frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}\}$ , then  $k(a) = \cos\left(\frac{\pi}{5}\right) + \cos\left(\frac{2\pi}{5}\right) \approx 1,1180$ .*
- *If  $a \in \{\frac{\pi}{8}, \frac{\pi}{4}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{5\pi}{4}, \frac{7\pi}{8}\}$ , then  $k(a) = \sqrt{2} \approx 1,4142$ .*
- *If  $a \in \{\frac{\pi}{11}, \frac{2\pi}{11}, \frac{3\pi}{11}, \frac{4\pi}{11}, \frac{5\pi}{11}, \frac{6\pi}{11}, \frac{7\pi}{11}, \frac{8\pi}{11}, \frac{9\pi}{11}, \frac{10\pi}{11}\}$ , then  $k(a) = \cos\left(\frac{2\pi}{11}\right) + \cos\left(\frac{3\pi}{11}\right) \approx 1,4961$ .*
- *If  $a \in \{0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{5\pi}{6}\} \cup \{\frac{\pi}{9}, \frac{2\pi}{9}, \frac{4\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}, \frac{8\pi}{9}\} \cup \{\frac{\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}\} \cup \{\frac{\pi}{15}, \frac{2\pi}{15}, \frac{4\pi}{15}, \frac{7\pi}{15}, \frac{8\pi}{15}, \frac{11\pi}{15}, \frac{13\pi}{15}, \frac{14\pi}{15}\}$ , then  $k(a) = 1.5$ .*
- *For all other values of  $a$ , we have  $1.5 < k(a) \leq \frac{8}{3\sqrt{3}} \approx 1.5396$ .*

**Corollary 3.15.** *Let  $G$  be an abelian group, and let  $(C(g))_{g \in G}$  be a  $G$ -cosine family in a unital Banach algebra  $A$  such that  $\sup_{g \in G} \|C(g) - c(g)\| < \frac{\sqrt{5}}{2}$  for some bounded scalar  $G$ -cosine family  $(c(g))_{g \in G}$ . Then  $C(g) = c(g)$  for every  $g \in G$ .*

Proof : Let  $g \in G$ . Since the scalar cosine sequence  $(c(ng))_{n \in \mathbb{Z}}$  is bounded, a standard argument shows that there exists  $a(g) \in \mathbb{R}$  such that  $c(ng) = \cos(na(g))1_A$  for  $n \in \mathbb{Z}$ . Since  $k(a(g)) \geq \frac{\sqrt{5}}{2}$ , it follows from corollary 2.4 that  $C(ng) = \cos(na(g))1_A = c(ng)$  for  $n \in \mathbb{Z}$ , and  $C(g) = c(g)$ .  $\square$

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